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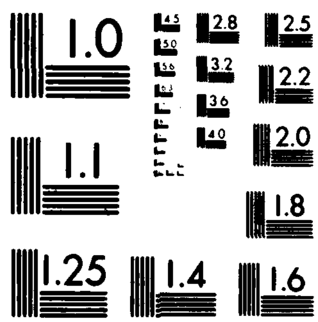

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THE NUMERICAL SOLUTION OF A CLASS OF CONSTRAINED MINIMIZATION PROBLEMS

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ABSTRACT

This paper proves that a large class of iterative schemes can be used to solve a certain constrained minimization problem. The constrained minimization problem considered involves the minimization of a quadratic functional subject to linear equality constraints. Among this class of convergent iterative schemes are generalizations of the relaxed Jacobi, Gauss-Seidel, and symmetric Gauss-Seidel schemes.

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SIGNIFICANCE AND EXPLANATION

Consider the following problem: Find the real n-vector x_* which minimizes $f(x) \equiv \frac{1}{2}x^TAx - x^Tr$ subject to the constraints $g(x) \equiv E^Tx - s = 0$. The purpose of this paper is to show that there is a large class of iterative schemes which can be used to solve such problems. These schemes are particularly effective when the matrix A is large and sparse and there are only a moderate number of constraints; one example of such a problem is described at the end of this paper.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

THE NUMERICAL SOLUTION OF A CLASS OF CONSTRAINED MINIMIZATION PROBLEMS

Nira Dyn* and Warren E. Ferguson**

§1. Introduction

In this paper we will present several iterative schemes which solve the following constrained minimization problem:

Problem 1: Find the real n-vector x_* which minimizes $f(x) \equiv \frac{1}{2} x^T Ax - x^T r$

subject to the constraints $g(x) \equiv E^T x - s = 0$.

Here A is a real symmetric nonnegative definite $n \times n$ matrix, E is a real $n \times m$ matrix with full column rank, r is a real n-vector, and s is a real m-vector.

As discussed in section 2, the theory of quadratic programming [Hadley] states that under reasonable conditions on A and E the solution of Problem 1 exists and is unique. Furthermore, under these conditions on A and E the solution x_* of Problem 1 is the x component of the solution (x_*,λ_*) of the following problem:

Problem 2: Find the real n-vector x_* and the real m-vector λ_* which

solves the linear system

$$\begin{bmatrix} A & E \\ E^T & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix} .$$

In section 3 we describe the convergence of a large class of iterative schemes used to solve Problem 2, and hence Problem 1. Although our iterative schemes are generally applicable to these problems, they are typically efficient only when A is a large sparse matrix and there are only a moderate number of constraints. In this situation the usual methods used to solve these problems become inefficient. The application of similar iterative schemes to the minimization of a quadratic form under inequality constraints is investigated in [Mangasarian].

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Our work was motivated by the work of [Tobler] in which a variant of one of the iterative schemes described in this paper was used to numerically construct a smooth surface from aggregated data. This application is described in section 4. The numerical solution of Problems 1 and 2 has also been considered by other authors; in particular we mention the work presented by [Luenberger], [Paige, Saunders], and [Gill, Murray].

§2. Preliminaries

In the previous section we stated that under reasonable conditions on A and E the solution x_* of Problem 1 is part of the solution (x_*,λ_*) of Problem 2. This statement is contained in the following theorem.

Theorem 2.1: Assume that

(a) A is a real symmetric nonnegative definite matrix,

(b) E is a real matrix with full column rank, and

(c) A and E^T have no nontrivial null vectors in common.

Then the solutions of Problems 1 and 2 exist and are unique. Furthermore: if x_* is the solution of Problem 1 then $(x_*, (E^TE)^{-1}E^T(r - Ax_*))$ is the solution of Problem 2, if (x_*,λ_*) is the solution of Problem 2 then x_* is the solution of Problem 1.

Proof: See the treatment of quadratic programming given in [Hadley]. ∎

Corollary 2.2: Under the assumptions of Theorem 2.1 the matrix

$$\begin{bmatrix} A & E \\ E^T & 0 \end{bmatrix}$$

is nonsingular.

Proof: Observe that this matrix is the coefficient matrix of the linear system in Problem 2. Under the assumptions of Theorem 2.1 this linear system has only unique solutions, therefore as shown in [Stewart] the coefficient matrix is nonsingular. ∎

The iterative schemes we use to solve Problem 2 are all based upon a splitting

$$A = B - C$$

of the matrix A, and they have the following form. Given an initial iterate (x_0,λ_0) define (x_k,λ_k) for $k = 1,2,3,\ldots$ to be the solution of the linear system

$$\begin{bmatrix} B & E \\ E^T & 0 \end{bmatrix} \begin{bmatrix} x_k \\ \lambda_k \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{k-1} \\ \lambda_{k-1} \end{bmatrix} + \begin{bmatrix} r \\ s \end{bmatrix} . \qquad (1)$$

Of course, for this linear stationary iterative method to be well defined it is necessary and sufficient that the matrix

$$\begin{bmatrix} B & E \\ E^T & 0 \end{bmatrix}$$

be nonsingular. This problem is addressed by the following theorem.

Theorem 2.3: In addition to the assumptions of Theorem 2.1 let

(d) $A = B - C$,

(e) B be a real nonsingular matrix, and

(f) $2A + C + C^T$ be a positive definite matrix.

Then the iterative scheme (1) is well-defined.

Proof: Since B is nonsingular then

$$\begin{bmatrix} B & E \\ E^T & 0 \end{bmatrix} = \begin{bmatrix} B & 0 \\ E^T & -E^T B^{-1} E \end{bmatrix} \begin{bmatrix} I & B^{-1}E \\ 0 & I \end{bmatrix} ,$$

therefore it follows that the matrix on the left hand side of the equality is nonsingular if and only if $E^T B^{-1} E$ is nonsingular. To prove that $E^T B^{-1} E$ is nonsingular let us prove that $E^T B^{-1} E$ has no nontrivial null vector. If $E^T B^{-1} E\lambda = 0$ then

$$\begin{aligned} 0 &= \lambda^T E^T B^{-1} E\lambda \\ &= (B^{-1}E\lambda)^T B^T (B^{-1}E\lambda) \\ &= \tfrac{1}{2} (B^{-1}E\lambda)^T (B + B^T)(B^{-1}E\lambda) \\ &= \tfrac{1}{2} (B^{-1}E\lambda)^T (2A + C + C^T)(B^{-1}E\lambda) \quad . \end{aligned}$$

This implies $B^{-1}E\lambda = 0$, since $2A + C + C^T$ is a positive definite matrix, and so $\lambda = 0$ because E is a matrix with full column rank.

∎

Let us now describe one procedure for solving the linear system (1) for (x_k, λ_k).

Step 1: Solve

$$B\hat{x}_k = Cx_{k-1} + r$$

for $\hat{x}_k$.

Step 2: Solve

$$(E^T B^{-1} E)\lambda_k = E^T \hat{x}_k - s$$

for λ_k.

Step 3: Solve

$$B(x_k - \hat{x}_k) = -E\lambda_k$$

for $x_k - \hat{x}_k$.

Step 2 requires the solution of a full linear system of order m. Therefore the iterative scheme is efficient only when $m \ll n$.

In the next section we will see that if assumption (f) of Theorem 2.3 is slightly strengthened then the iterative scheme (1) is not only well-defined but also convergent.

§3. Convergence of the Iterative Schemes

One set of conditions which guarantees the convergence of the iterative scheme (1) is described in the following theorem.

Theorem 3.1: Assume that

(a) A is a real symmetric nonnegative definite matrix,

(b) E is a real matrix with full column rank,

(c) A and E^T have no nontrivial null vectors in common,

(d) $A = B - C$,

(e) B is a real nonsingular matrix, and

(f) $A + C + C^T$ is a positive definite matrix.

Then the iterative scheme (1) is well-defined and convergent.

Proof: From Theorem 2.1 we deduce that a solution of Problem 2 exists and is unique. Since A is a nonnegative definite matrix and $A + C + C^T$ is a positive definite matrix then $2A + C + C^T$ is a positive definite matrix. From Theorem 2.3 we therefore deduce that the iterative scheme (1) is well-defined.

As shown in [Wendroff], the iterative scheme (1) is convergent if and only if each eigenvalue of the matrix

$$\begin{bmatrix} B & E \\ E^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \tag{2}$$

has magnitude less than one. Let us therefore show that if μ is a nonzero eigenvalue of the matrix (2) then the magnitude of μ is less than one.

Since μ is an eigenvalue of the matrix (2) then there are complex vectors u,v not both zero for which

$$\mu \begin{bmatrix} B & E \\ E^T & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} . \tag{3}$$

Let us now argue that $u \neq 0$ and $u^H Au > 0$. Since A is a real symmetric nonnegative definite matrix then clearly $u^H Au \geq 0$, and $u^H Au = 0$ only if $Au = 0$.

However $Au = 0$ only if $u = 0$, for (3) states that $E^T u = 0$ and by hypothesis A and E^T have no nontrivial null vectors in common. But $u = 0$ only if $v = 0$, for by hypothesis E has full column rank and (3) implies that $Ev = 0$ when $u = 0$. Since u,v are not both zero then we conclude $u \neq 0$ and $u^H Au > 0$.

Let us now establish the fact that

$$(1 - |\mu|^2)u^H Au = |1 - \mu|^2 u^H (A + C + C^T)u \quad . \tag{4}$$

We begin with the identity

$$u^H Au - (\mu u)^H A(\mu u) = (u - \mu u)^H A(u - \mu u) + 2\mathrm{Re}\{(u - \mu u)^H A(\mu u)\} \quad . \tag{5}$$

Using (3), and the fact that $A = B - C$, we find that

$$\begin{aligned}(u - \mu u)^H A(\mu u) &= (u - \mu u)^H (B-C)(\mu u)\\ &= (u - \mu u)^H (\mu Bu - \mu Cu)\\ &= (u - \mu u)^H (Cu - \mu Ev - \mu Cu)\\ &= (u - \mu u)^H C(u - \mu u) \quad ,\end{aligned}$$

which reduces (5) to the result stated in (4).

By hypothesis, $A + C + C^T$ is a positive definite matrix. Since $u \neq 0$ we know that $u^H(A + C + C^T)u > 0$, and so (4) implies that either $|\mu| < 1$ or $\mu = 1$. However it is impossible that $\mu = 1$, for if $\mu = 1$ then (3) would imply that u,v are both zero because by Corollary 2.2 the matrix

$$\begin{bmatrix} A & E \\ E^T & 0 \end{bmatrix}$$

is nonsingular.

■

Let us now describe several iterative schemes whose convergence is assured by Theorem 3.1.

Corollary 3.2: Assume that

(a) A is a real symmetric nonnegative definite matrix,

(b) E is a real matrix with full column rank,

(c) A and E^T have no nontrivial null vectors in common,

(g) $A = D - L - L^T$ where D is a nonsingular diagonal matrix and L is a strictly lower triangular matrix.

Then the iterative scheme (1) is convergent for the following choices of B and C.

(1) $B = \frac{1}{\omega} D$, $C = (\frac{1-\omega}{\omega})D + L + L^T$ with $\omega > 0$ chosen so small that $\frac{2}{\omega} D - A$ is a positive definite matrix.

(2) $B = \frac{1}{\omega} D - L$, $C = \frac{1-\omega}{\omega} D + L^T$ with $0 < \omega < 2$.

(3) $B = (\frac{2-\omega}{\omega})^{-1}(\frac{1}{\omega} D - L)D^{-1}(\frac{1}{\omega} D - L)^T$,

$C = (\frac{2-\omega}{\omega})^{-1}(\frac{1-\omega}{\omega} D + L)D^{-1}(\frac{1-\omega}{\omega} D + L)^T$ with $0 < \omega < 2$.

Proof: It is clear that assumptions (a) thru (e) of Theorem 3.1 are valid for each of the above choices for B and C. Therefore the iterative scheme (1) will be convergent if assumption (f) of Theorem 3.1 is valid for each of the above choices of B and C. For the first choice of B and C we find that

$$A + C + C^T = \frac{2}{\omega} D - A$$

and so assumption (f) of Theorem 3.1 is valid if $\omega > 0$ is chosen so small that $\frac{2}{\omega} D - A$ is positive definite. For the second choice of B and C we find that

$$A + C + C^T = \frac{2-\omega}{\omega} D$$

and so assumption (f) of Theorem 3.1 is valid if $0 < \omega < 2$. For the third choice of B and C we note that

$$A + C + C^T = B + C \quad ,$$

where $B(C)$ is a real symmetric positive (non-negative) definite matrix if $0 < \omega < 2$, and so assumption (f) of Theorem 3.1 is valid if $0 < \omega < 2$.

■

We note that the first, second, and third choices of B and C described in Corollary 3.2 correspond respectively to the usual JOR, SOR, and SSOR splittings of A described in [Young]. Under further assumptions Corollary 3.2 can be extended to the line and block versions of JOR, SOR, and SSOR. Furthermore, there is an obvious generalization of Theorem 3.1 to complex matrices.

§4. Application of the Iterative Scheme

Let us consider the problem of estimating a certain geographically varying quantity over a bounded geographical region, given the integrals of the quantity over several disjoint sub-regions.

Let Ω represent this finite geographical region and $\Omega_1, \Omega_2, \ldots, \Omega_m$ its disjoint sub-regions. In addition let $v(x,y)$ represent the function over Ω whose values we wish to estimate.

We shall choose the function $u^*(x,y)$ by requiring that it minimize the functional

$$J(u) = \tfrac{1}{2} \int_\Omega |\nabla u(x,y)|^2 dxdy \tag{5}$$

among all functions in the Sobolev space $H^1(\Omega)$ which satisfy the constraints

$$\int_{\Omega_i} u(x,y) dxdy = s_i \quad \text{for} \quad i = 1,2,\ldots,m \quad . \tag{6}$$

Here the numbers $s_1, s_2, \ldots, s_m$ are the given values of the integral of $v(x,y)$ over the subregions $\Omega_1, \Omega_2, \ldots, \Omega_m$. This problem was considered by [Tobler] and analyzed in [Dyn, Wahba]; in particular the paper by [Dyn, Wahba] demonstrates that this problem is well-posed.

This problem is discretized by the finite element method as described in [Friedrichs, Keller]. Here a triangulation Ω_T of Ω is introduced and $u^*(x,y)$ is approximated by the continuous function $u(x,y)$ which is a linear function of x and y in each triangle of Ω_T, minimizes the functional (5), and satisfies the constraints (6).

Suppose that the vertices $P_1, P_2, \ldots, P_n$ of the triangles in Ω_T are labeled in some order. If the i-th component of the n-vector u represents the value of $u(x,y)$ at P_i then the functional (5) admits that representation

$$J(u) = \tfrac{1}{2} u^T A u$$

where A is a real symmetric nonnegative definite $n \times n$ matrix having positive diagonal entries. In addition the constraints (6) admit the representation

$$E^T u = s$$

where E is a real $n \times m$ matrix and s is a real m-vector whose i-th component is s_i. If the triangulation Ω_T is sufficiently fine then the matrix E will have full

column rank. Therefore the n-vector u can be determined by any of the iterative schemes described in Corollary 3.2.

The convergence of the solution u to the solution u^* of problem (5) and (6), as the triangulation become finer, is demonstrated in [Dyn, Ferguson].

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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